



Martingales and Measures

Chapter 27

Derivatives Dependent on a Single Underlying Variable

Consider a variable θ (not necessarily the price of a traded security) that follows the process

$$\frac{d\theta}{\theta} = m dt + s dz$$

Imagine two derivatives dependent on θ with prices f_1 and f_2 . Suppose

$$\frac{df_1}{f_1} = \mu_1 dt + \sigma_1 dz$$

$$\frac{df_2}{f_2} = \mu_2 dt + \sigma_2 dz$$

Forming a Riskless Portfolio

We can set up a riskless portfolio Π consisting of

+ $\sigma_2 f_2$ of the 1st derivative and
- $\sigma_1 f_1$ of the 2nd derivative

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$$

$$\Delta \Pi = (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2) \Delta t$$

Market Price of Risk (Page 624)

Since the portfolio is riskless $\Delta \Pi = r \Pi \Delta t$

This gives: $\mu_1 \sigma_2 - \mu_2 \sigma_1 = r \sigma_2 - r \sigma_1$

$$\text{or } \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

- This shows that $(\mu - r)/\sigma$ is the same for all derivatives dependent on the same underlying variable, θ
- We refer to $(\mu - r)/\sigma$ as the market price of risk for θ and denote it by λ

Extension of the Analysis to Several Underlying Variables

(Equations 27.12 and 27.13, page 627)

If f depends on several underlying variables
with

$$\frac{df}{f} = \mu dt + \sum_{i=1}^n \sigma_i dz_i$$

then

$$\mu - r = \sum_{i=1}^n \lambda_i \sigma_i$$

Martingales (Page 628)

- A martingale is a stochastic process with zero drift
- A variable following a martingale has the property that its expected future value equals its value today

Alternative Worlds

In the tradition of risk-neutral world

$$df = rf dt + \sigma f dz$$

In a world where the market price of risk is λ

$$df = (r + \lambda \sigma) f dt + \sigma f dz$$

The Equivalent Martingale Measure Result (Page 628-629)

If we set λ equal to the volatility of a security g , then to's lemma shows that f/g is a martingale for all derivative security prices f

Forward Risk Neutrality

We will refer to a world where the market price of risk is the volatility of g as a world that is forward risk neutral with respect to g .

If E_g denotes a world that is FRN wrt g

$$\frac{f_0}{g_0} = E_g \left(\frac{f_T}{g_T} \right)$$

Alternative Choices for the Numeraire Security g

- Money Market Account
- Zero-coupon bond price
- Annuity factor

Money Market Account as the Numeraire

- The money market account is an account that starts at \$1 and is always invested at the short-term risk-free interest rate
- The process for the value of the account is
$$dg = rg \, dt$$
- This has zero volatility. Using the money market account as the numeraire leads to the traditional risk-neutral world where $\lambda = 0$

Money Market Account continued

Since $g_0 = 1$ and $g_T = e^{\int_0^T r dt}$, the equation

$$\frac{f_0}{g_0} = E_g \left(\frac{f_T}{g_T} \right)$$

becomes

$$f_0 = \hat{E} \left[e^{-\int_0^T r dt} f_T \right]$$

where $\hat{E}$ denotes expectations in the
tradition risk-neutral world

Zero-Coupon Bond Maturing at time T as Numeraire

The equation

$$\frac{f_0}{g_0} = E_g \left(\frac{f_T}{g_T} \right)$$

becomes

$$f_0 = P(0, T) E_T [f_T]$$

where $P(0, T)$ is the zero-coupon bond price
and E_T denotes expectations in a world that
is FRN wrt the bond price

Forward Prices

In a world that is FRN wrt $P(0, T)$, the expected value of a security at time T is its forward price

Interest Rates

In a world that is FRN wrt $P(0, T_2)$ the expected value of an interest rate lasting between times T_1 and T_2 is the forward interest rate

Annuity Factor as the Numeraire

The equation

$$\frac{f_0}{g_0} = E_g \left(\frac{f_T}{g_T} \right)$$

becomes

$$f_0 = A(0) E_A \left[\frac{f_T}{A(T)} \right]$$

Annuity Factors and Swap Rates

Suppose that $s(t)$ is the swap rate corresponding to the annuity factor A . Then:

$$s(t) = E_A[s(T)]$$

Extension to Several Independent Factors (Page 633)

In the traditional risk-neutral world

$$df(t) = r(t) f(t)dt + \sum_{i=1}^m \sigma_{f,i}(t) f(t) dz_i$$

$$dg(t) = r(t) g(t)dt + \sum_{i=1}^m \sigma_{g,i}(t) g(t) dz_i$$

For other worlds that are internally consistent

$$df(t) = \left[r(t) + \sum_{i=1}^m \lambda_i \sigma_{f,i}(t) \right] f(t)dt + \sum_{i=1}^m \sigma_{f,i}(t) f(t) dz_i$$

$$dg(t) = \left[r(t) + \sum_{i=1}^m \lambda_i \sigma_{g,i}(t) \right] g(t)dt + \sum_{i=1}^m \sigma_{g,i}(t) g(t) dz_i$$

Extension to Several Independent Factors continued

We define a world that is FRN wrt g as world where $\lambda_i = \sigma_{g,i}$

As in the one-factor case, f/g is a martingale and the rest of the result hold.

Applications

- Extension of Black's model to case where interest rates are stochastic
- Valuation of an option to exchange one asset for another

Black's Model (page 634)

By working in a world that is forward risk neutral with respect to a $P(0, T)$ it can be seen that Black's model is true when interest rates are stochastic providing the forward price of the underlying asset is has a constant volatility

$$c = P(0, T)[F_0 N(d_1) - KN(d_2)]$$

$$p = P(0, T)[KN(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0/K) + \sigma_F^2 T}{\sigma_F \sqrt{T}} \quad d_2 = \frac{\ln(F_0/K) - \sigma_F^2 T}{\sigma_F \sqrt{T}}$$

Option to exchange an asset worth U for one worth V

- This can be valued by working in a world that is forward risk neutral with respect to U
- See equations 27.30, 27.31, and 27.32

Change of Numeraire

(Section 27.8, page 636)

When we change the numeraire security from g to h , the drift of a variable v increases by $\rho \sigma_v \sigma_w$ where σ_v is the volatility of v , $w = h/g$, σ_w is the volatility of w , and ρ is the correlation between v and w .